

A Poincaré-type inequality and a related eigenvalue problem

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Abstract

Given a smooth positive function f defined on the unit circle satisfying a simple condition, we obtain a Poincaré-type inequality for an arbitrary function u whose weighted average with respect to f is zero. The proof uses Fenchel's theorem about the total curvature of closed space curves in an essential way. Next we consider the generalization of this result to higher dimensional closed Riemannian manifold and reduce it to an eigenvalue problem. Finally, we point out that even though such Poincaré-type inequality still holds, the best constant $\lambda_1(f)$ might be different from the first eigenvalue λ_1 by constructing explicit examples on the standard spheres and flat tori.

Keywords: Poincaré-type inequality, Fenchel's theorem, f -eigenvalue, interlace theorem.

MSC(2000):

1 Introduction

In this paper, we report a Poincaré-type inequality for functions defined on S^1 as below.

Theorem 1.1. *Let $f(\theta)$ be a fixed positive 2π -period C^1 function on \mathbb{R} such that*

$$\int_0^{2\pi} f(\theta) \cos \theta d\theta = \int_0^{2\pi} f(\theta) \sin \theta d\theta = 0. \quad (1)$$

Then for any 2π -period C^1 function ϕ satisfying

$$\int_0^{2\pi} f(\theta) \phi(\theta) d\theta = 0, \quad (2)$$

we have

$$\int_0^{2\pi} \phi(\theta)^2 d\theta \leq \int_0^{2\pi} (\phi'(\theta))^2 d\theta. \quad (3)$$

The equality is attained only when $\phi(\theta) = a \cos \theta + b \sin \theta$ for real constants a, b .

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To the best of our knowledge, this inequality seems to be a new result. It is an unexpected byproduct of our study of the global geometry of closed curves in 3-dimensional (Euclidean and Lorentz) space, and the proof follows from the famous Fenchel's theorem (see Section 2 and [7]).

We tried to find a purely analytical proof without success. This situation raises two tantalizing questions: What is the significance of this analytical result? Can we generalize it to functions on other higher dimensional Riemannian manifolds?

Notice that if we take $f \equiv 1$ identically in the theorem above, then (1) holds true automatically, and our result reduces to the classical Wirtinger's inequality. The natural generalization of the latter to higher dimensional spaces is Poincaré inequality [8], which states that for a given closed Riemannian manifold (M^n, g) , there exists a universal constant $C > 0$ such that for any C^1 functions u satisfying

$$\int_M u d\mu = 0, \quad (4)$$

there always holds

$$C \int_M u^2 d\mu \leq \int_M |\nabla u|^2 d\mu. \quad (5)$$

In particular, C can be chosen to be λ_1 , the first eigenvalue of the standard Laplacian on (M^n, g) .

Based on this observation, we propose the following conjecture as a parallel generalization of our Poincaré-type inequality (3):

Conjecture 1.2. *Let (M^n, g) be a compact Riemannian manifold with $\partial M = \emptyset$. Let λ_1 be the first eigenvalue of the Laplace operator Δ on (M^n, g) and \mathcal{U}_1 be the first eigenspace of Δ . Let $f : M \rightarrow \mathbb{R}^+$ be a fixed smooth function satisfying $f \perp \mathcal{U}_1$. Then for any smooth function $\phi : M \rightarrow \mathbb{R}$ satisfying $\int_M f \phi d\mu = 0$, we have*

$$\lambda_1 \int_M \phi^2 d\mu \leq \int_M |\nabla \phi|^2 d\mu. \quad (6)$$

If we denote the Rayleigh quotient by $R(u) = \int |\nabla u|^2 d\mu / \int u^2 d\mu$, then this problem is equivalent to showing

$$\lambda_1 \leq \inf_{u \in f^\perp} R(u). \quad (7)$$

The right hand side could be viewed as a constrained variation problem, whose corresponding Euler-Lagrange equation is

$$\Delta u + R(u)u \equiv 0 \pmod{f}. \quad (8)$$

Thus we make the following definitions.

Definition 1.3. *Let (M^n, g) be a closed Riemannian manifold with a fixed positive function f defined on it. A constant λ is called an f -eigenvalue if there exist some non-trivial smooth function $u \in f^\perp$ (with respect to the L^2 norm on the function space over (M^n, g)) such that*

$$\Delta u \equiv -\lambda u \pmod{f}. \quad (9)$$

This corresponding u is called the related f -eigenfunction.

As usual, such f -eigenvalues form a spectrum $\{\lambda_k(f)\}_{k=1}^{+\infty}$, called *the f -spectrum*:

$$0 \leq \lambda_1(f) \leq \lambda_2(f) \leq \cdots \leq \lambda_k(f) \leq \cdots \quad (10)$$

They share many similar properties as the usual eigenvalues $\{\lambda_k\}_{k=1}^{+\infty}$ [1]. In particular, the min-max principle still holds true. This enables us to generalize the classical Cauchy interlace theorem directly (which is also known as Poincaré separation theorem; for reference, please see [4, 5, 6]). As part of the results, generally we have

$$\lambda_1(f) = \inf_{u \in f^\perp} R(u) \leq \lambda_1.$$

We observe that:

- 1) $f \perp \mathcal{U}_1$ is a necessary condition for $\lambda_1(f) = \lambda_1$.
- 2) Conjecture 1.2 means that $f \perp \mathcal{U}_1$ is also sufficient to derive $\lambda_1(f) = \lambda_1$.

Although we obtained the interlace theorem and some other results, our attempt to prove Conjecture 1.2 all failed. Finally this conjecture turns out to be not true.

Conclusion: There exist counterexamples to Conjecture 1.2 on the spheres $S^n (n \geq 2)$ (with the standard metric) and on the square torus T^2 (with the flat metric).

The basic idea for the construction of these counterexamples is:

1. Choose M from the simplest Riemannian manifolds with many symmetries whose eigenfunctions and eigenvalues are known explicitly.
2. Using the Ansatz that $\phi = \phi_2 + \epsilon$, i.e., a combination of some second eigenfunction ϕ_2 and the constant function. The constant ϵ is chosen suitably so that ϕ changes sign on M , at the same time the Rayleigh quotient $R(\phi) = \int |\nabla \phi|^2 d\mu / \int \phi^2 d\mu$ is smaller than λ_1 .
3. Show that there exists a positive even function f orthogonal to the first eigenspace \mathcal{U}_1 (which consist of only odd functions) and ϕ (which changes its sign).

Note that these counterexamples are only with respect to some special functions f . If the function f are given arbitrarily, the answer might be different. For example, the Poincaré-type inequality is still true when f is a very small perturbation of the constant function. In view of these facts, the following problem seems to be subtle and interesting.

Problem: Is there any closed Riemannian manifold (M^n, g) other than S^1 such that the conclusion of Conjecture 1.2 holds true for any positive function $f \perp \mathcal{U}_1$?

We guess the answer is negative. If so, then S^1 can be characterized as the unique Riemannian manifold on which the generalized Poincaré inequality (6) holds true under the weakest conditions in Conjecture 1.2.

This paper is organized as follows. In Section 2 we prove Theorem 1.1. The formulation of the eigenvalue problem and the estimation of the f -eigenvalues are presented in Section 3. We give the construction of the counterexamples to Conjecture 1.2 in the last section.

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2 The proof to the main theorem

This section establishes the Poincaré type inequality for functions on S^1 . The inequality is obtained as a corollary of Fenchel's theorem [2, 3] on the total curvature of closed curves in \mathbb{R}^3 . Then we will characterize the equality case in the second subsection.

2.1 The inequality follows from Fenchel's theorem

By the assumption of Theorem 1.1, we are given a fixed 2π -period C^1 function $f(\theta)$ on \mathbb{R} satisfying: 1) $\int_0^{2\pi} f(\theta) \cos \theta d\theta = \int_0^{2\pi} f(\theta) \sin \theta d\theta = 0$; 2) f is a positive function. Set

$$\gamma_\sigma(\theta) = \int_0^\theta f(\theta)(\cos \theta, \sin \theta, \sigma \phi(\theta)) d\theta, \theta \in [0, 2\pi], \sigma \in (-\epsilon, \epsilon). \quad (11)$$

Then γ_σ form an one-parameter family of C^2 closed curves in \mathbb{R}^3 (which differ from each other by a vertical scaling). By standard computation we find their total curvatures as

$$L(\sigma) \triangleq \int_{\gamma_\sigma} k ds = \int_0^{2\pi} \frac{\sqrt{1 + \sigma^2 \phi^2 + \sigma^2 (\phi')^2}}{1 + \sigma^2 \phi^2} d\theta. \quad (12)$$

Fenchel's theroem says $L(\sigma) \geq 2\pi = L(0)$. Thus there should hold $L'(0) = 0$ (which is obviously true for our γ_σ), and $L''(0) \geq 0$. A direct computation shows that

$$L''(0) = \int_0^{2\pi} (\phi'(\theta))^2 d\theta - \int_0^{2\pi} \phi(\theta)^2 d\theta, \quad (13)$$

so the inequality follows immediately.

2.2 The equality case

We have verified the inequality as above. To determine the equality case, which is more involved and subtler, we need to use this inequality to prove two technical lemmas beforehand.

Before giving the statement of the lemmas, we observe that the positivity of f is a crucial assumption in Theorem 1.1. Without this assumption, the orthogonality condition $f \perp \mathcal{U}_1$ is not sufficient to establish the inequality (3). (An obvious counterexample is provided by choosing $f = \phi_2$ as the second eigenfunction and $\phi \equiv 1$.) Geometrically, the assumption $f > 0$ guarantees that θ is a regular parameter of the closed space curves γ_σ as above. Analytically, this requires that in the Fourier series of f , the first coefficient should not be too small compared with other coefficients. The following two lemmas provide a quantitative description of this restriction.

Lemma 2.1. *Let $f(\theta)$ be a fixed 2π -period C^1 function on \mathbb{R} with*

$$\int_0^{2\pi} f(\theta) \cos \theta d\theta = \int_0^{2\pi} f(\theta) \sin \theta d\theta = 0. \quad (14)$$

Write $f(\theta) = \frac{a_0}{2} + \sum_{n=2}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)$. Suppose f is positive. Then

$$\sum_{n=2}^{\infty} \frac{a_n^2 + b_n^2}{n^2 - 1} \leq \frac{a_0^2}{2}. \quad (15)$$

Proof. Consider an arbitrary 2π -period C^1 function $\phi(\theta)$ orthogonal to f and write

$$\phi(\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} (c_n \cos n\theta + d_n \sin n\theta). \quad (16)$$

The orthogonality implies

$$c_0 = -\frac{2}{a_0} \sum_{n=2}^{\infty} (a_n c_n + b_n d_n). \quad (17)$$

By the inequality of Theorem 1.1 established in the previous subsection, there follows

$$\int_0^{2\pi} \phi(\theta)^2 d\theta \leq \int_0^{2\pi} (\phi'(\theta))^2 d\theta. \quad (18)$$

Substituting (16) and (17) into (18), we obtain

$$\frac{2}{a_0^2} \left[\sum_{n=2}^{\infty} (a_n c_n + b_n d_n) \right]^2 = \frac{c_0^2}{2} \leq \sum_{n=2}^{\infty} (n^2 - 1)(c_n^2 + d_n^2). \quad (19)$$

Consider the following two vectors in the ℓ^2 space:

$$\xi = \left(\frac{a_2}{\sqrt{2^2 - 1}}, \frac{b_2}{\sqrt{2^2 - 1}}, \dots, \frac{a_n}{\sqrt{n^2 - 1}}, \frac{b_n}{\sqrt{n^2 - 1}}, \dots \right), \quad (20)$$

$$\eta = (\sqrt{2^2 - 1}c_2, \sqrt{2^2 - 1}d_2, \dots, \sqrt{n^2 - 1}c_n, \sqrt{n^2 - 1}d_n, \dots). \quad (21)$$

Then (19) is equivalent to

$$\frac{\langle \xi, \eta \rangle^2}{|\eta|^2} \leq \frac{a_0^2}{2} \quad (22)$$

for arbitrary nonzero $\eta \in \ell^2$. On the other hand, the Cauchy-Schwarz inequality implies

$$\sup_{\eta \neq 0} \frac{\langle \xi, \eta \rangle^2}{|\eta|^2} = |\xi|^2 = \sum_{n=2}^{\infty} \frac{a_n^2 + b_n^2}{n^2 - 1}. \quad (23)$$

The conclusion follows directly from (22) and (23). \square

Lemma 2.2. *Under the same assumption as Lemma 2.1, we have*

$$\sum_{n=2}^{\infty} \frac{a_n^2 + b_n^2}{n^2 - 1} < \frac{a_0^2}{2}. \quad (24)$$

Proof. As a positive C^1 function, f has a positive minimal value. So $f(\theta) - \frac{\epsilon}{2}$ is also a positive 2π -period C^1 function for ϵ small enough. Applying Lemma 2.1 to $\tilde{f}(\theta) = f(\theta) - \frac{\epsilon}{2} = \frac{a_0 - \epsilon}{2} + \sum_{n=2}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)$ instead of $f(\theta)$, we know

$$\sum_{n=2}^{\infty} \frac{a_n^2 + b_n^2}{n^2 - 1} \leq \frac{(a_0 - \epsilon)^2}{2} < \frac{a_0^2}{2}. \quad \square$$

Proof of Theorem 1.1. After establishing the inequality in Subsection 2.1, here we need only to determine the equality case. Assume that the equality holds true in (3) for some function $\phi(\theta)$. Write out the Fourier series:

$$\phi(\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} (c_n \cos n\theta + d_n \sin n\theta), \quad (25)$$

$$f(\theta) = \frac{a_0}{2} + \sum_{n=2}^{\infty} (a_n \cos n\theta + b_n \sin n\theta). \quad (26)$$

Then

$$\int_0^{2\pi} f(\theta)\phi(\theta)d\theta = 0 \Leftrightarrow c_0 = -\frac{2}{a_0} \sum_{n=2}^{\infty} (a_n c_n + b_n d_n), \quad (27)$$

$$\int_0^{2\pi} \phi(\theta)^2 d\theta = \int_0^{2\pi} (\phi'(\theta))^2 d\theta \Leftrightarrow \sum_{n=2}^{\infty} (n^2 - 1)(c_n^2 + d_n^2) = \frac{c_0^2}{2}. \quad (28)$$

By (27) and (28), we have

$$\frac{a_0^2}{2} \sum_{n=2}^{\infty} (n^2 - 1)(c_n^2 + d_n^2) = \left[\sum_{n=2}^{\infty} (a_n c_n + b_n d_n) \right]^2. \quad (29)$$

Using the same notations as in (20) and (21), and invoking the Cauchy-Schwarz inequality, from (29) we obtain

$$\frac{a_0^2}{2} |\eta|^2 = \langle \xi, \eta \rangle^2 \leq |\xi|^2 |\eta|^2. \quad (30)$$

On the other hand, we also have $|\xi|^2 < \frac{a_0^2}{2}$ by Lemma 2.2. So there must be $\eta = 0$, i.e. $c_n = d_n = 0$ for all $n \geq 2$. Hence $c_0 = 0$ by (27). Finally we have $\phi(\theta) = c_1 \cos \theta + d_1 \sin \theta$ as desired. \square

Remark 2.3. It is interesting to note that the condition of the equality can be deduced from the inequality (3) itself. The key observation is that the stronger inequality (24) can be deduced from the weaker (15) by allowing a small perturbation of f to $f - \frac{\epsilon}{2}$.

Conjecture 1.2 is a generalization of the results in this section to any closed Riemannian manifold M . The orthogonality assumptions $f \perp \cos \theta, \sin \theta$ and $f \perp \phi$ enable us to recover closed space curves in \mathbb{R}^3 when $M = S^1$. But for other Riemannian manifolds we lack such a geometric picture. Neither can we find a connection with geometric quantities like the total curvature. So we need new method to solve this problem. The next section is such an attempt.

3 The f -eigenvalues and its estimation

In this section we fix a closed Riemannian manifold M and a positive function $f : M \rightarrow \mathbb{R}^+$. In the introduction we have noticed that Conjecture 1.2 is essentially about finding out the infimum of the Rayleigh quotient $R(u)$ where the function u is restricted on the orthogonal complement of f in the function space. Suppose

$u \in f^\perp$ is a critical point of this variational problem. Then for arbitrary $v \in f^\perp$, the v -variation of $R(u)$ is zero. By direct computation, that means

$$\begin{aligned} 0 = \delta_v R(u) &= \left. \frac{d}{ds} \right|_{s=0} R(u + sv) = \left. \frac{d}{ds} \right|_{s=0} \frac{\int_M |\nabla u + s \nabla v|^2 d\mu}{\int_M (u + sv)^2 d\mu} \\ &= \frac{2 \int_M \nabla u \cdot \nabla v d\mu}{\int_M u^2 d\mu} - 2 \frac{\int_M |\nabla u|^2 d\mu}{(\int_M u^2 d\mu)^2} \int_M u v d\mu \\ &= -\frac{2}{\int_M u^2 d\mu} \int_M [\Delta u + R(u)u] v d\mu, \quad \forall v \in f^\perp. \end{aligned}$$

So the Euler-Lagrange equation is $\Delta u + R(u)u = cf$ for some real constant c . This justifies Definition 1.3 given in the introduction, where the f -eigenvalue λ and the f -eigenfunction u is defined by the equality

$$\Delta u \equiv -\lambda u \pmod{f}. \quad (31)$$

Proposition 3.1. *Let $u_1, u_2 \in f^\perp$ be two f -eigenfunctions on M with distinct f -eigenvalues $\lambda_1 \neq \lambda_2$. Then u_1, u_2 are orthogonal to each other, i.e., $\int_M u_1 u_2 d\mu = 0$.*

Proof. By the definition of the f -eigenfunctions, $0 = \int_M \Delta u_1 u_2 d\mu - \int_M u_1 \Delta u_2 d\mu = (\lambda_2 - \lambda_1) \int_M u_1 u_2 d\mu$. \square

We can also define the k -th f -eigenvalues recursively by min-max method.

Definition 3.2. *Fix a closed Riemannian manifold M and a positive function $f : M \rightarrow \mathbb{R}^+$. Set*

$$\lambda_1(f) = \inf_{u \in f^\perp} R(u), \quad (32)$$

and let $\phi_{1,f}$ be the corresponding f -eigenfunction. Suppose that $\{\lambda_i(f)\}_{i=1}^k$ have been defined, and $\{\phi_{i,f}\}_{i=1}^k$ are the corresponding f -eigenfunctions. Then set

$$\lambda_{k+1}(f) = \inf_{u \in \text{Span}\{f, \{\phi_{i,f}\}_{i=1}^k\}^\perp} R(u). \quad (33)$$

Let $\phi_{k+1,f} \in \text{Span}\{f, \{\phi_{i,f}\}_{i=1}^k\}^\perp \neq 0$ to be a nonzero minimizer. This defines the f -Spectrum:

$$0 \leq \lambda_1(f) \leq \lambda_2(f) \leq \cdots \leq \lambda_k(f) \leq \cdots \quad (34)$$

Obviously, the f -eigenvalues are exactly the usual eigenvalues when f is a constant. In the general case, we want to know the relationships between these two kinds of eigenvalues.

Proposition 3.3. *Under the assumptions and notations as above, we have*

$$\lambda_1(f) \geq \lambda_1 \frac{(\int_M f d\mu)^2}{\int_M d\mu \int_M f^2 d\mu} > 0. \quad (35)$$

Proof. Denote f as $f = f_0 + \tilde{f}$, where $f_0 = \frac{1}{|M|} \int_M f d\mu > 0$ and $\int_M \tilde{f} d\mu = 0$.

For any $u \perp f$, written $u = u_0 + \tilde{u}$, where $u_0 = \frac{1}{|M|} \int_M u d\mu$. Then $u \perp f$ implies

$$\int_M \tilde{f} \tilde{u} d\mu = -f_0 \int_M u d\mu. \quad (36)$$

By Cauchy-Schwarz inequality, we have

$$f_0^2|M|(\int_M u^2 d\mu - \int_M \tilde{u}^2 d\mu) = f_0^2[\int_M u d\mu]^2 = [\int_M \tilde{f} \tilde{u} d\mu]^2 \leq \int_M \tilde{f}^2 d\mu \int_M \tilde{u}^2 d\mu. \quad (37)$$

Since $\int_M \tilde{u} d\mu = 0$, it gives $\lambda_1 \int_M \tilde{u}^2 d\mu \leq \int_M |\nabla \tilde{u}|^2 d\mu$ by definition. Then

$$f_0^2|M| \int_M u^2 d\mu \leq \frac{1}{\lambda_1} \int_M f^2 d\mu \int_M |\nabla \tilde{u}|^2 d\mu = \frac{1}{\lambda_1} \int_M f^2 d\mu \int_M |\nabla u|^2 d\mu. \quad (38)$$

It shows that $R(u) \geq \lambda_1 \frac{(\int_M f d\mu)^2}{|M| \int_M f^2 d\mu}$ for any $u \in f^\perp$, then the desired result is given immediately by taking the infimum on $R(u)$. \square

Proposition 3.4. *Under the assumptions and notations as the above definition, we have*

$$\lambda_1(f) \leq \lambda_1. \quad (39)$$

Denote \mathcal{U}_1 as the first eigenspace of the Laplace operator (here the first eigenvalue λ_1 might have multiplicity larger than 1). Then $\lambda_1(f) < \lambda_1$ if f is NOT orthogonal to \mathcal{U}_1 .

Proof. Let ϕ_1 be an eigenfunction of the first eigenvalue λ_1 . Then there exists some constant c such that $(\phi_1 + c) \perp f$. Note that

$$R(\phi_1 + c) = \frac{\int_M |\nabla \phi_1|^2 d\mu}{\int_M (\phi_1 + c)^2 d\mu} = \frac{\lambda_1 \int_M \phi_1^2 d\mu}{\int_M \phi_1^2 d\mu + c^2 |M|} \leq \lambda_1. \quad (40)$$

So we have

$$\lambda_1(f) = \inf_{u \in f^\perp} R(u) \leq R(\phi_1 + c) \leq \lambda_1. \quad (41)$$

If f is not orthogonal to \mathcal{U}_1 , then there exists an eigenfunction ϕ_1 such that $\int_M f \phi_1 d\mu \neq 0$. Then $c = -\frac{\int_M f \phi_1 d\mu}{\int_M f d\mu} \neq 0$. By (40) we know $\lambda_1(f) < \lambda_1$. \square

Proposition 3.5. *Let $\lambda_k(f)$ be the k -th f -eigenvalue, and λ_k be the k -th eigenvalue of the Laplacian. Then we have*

$$\lambda_k(f) \leq \lambda_k. \quad (42)$$

Proof. Let $\{\phi_i\}_{i=1}^k$ be the first k eigenfunctions (we do not count their multiplicities). Then there exist suitable non-trivial constants $\{c_i\}_{i=0}^k$ such that

$$(c_0 + \sum_{i=1}^k c_i \phi_i) \perp \text{Span}\{f, \{\phi_{i,f}\}_{i=1}^{k-1}\}.$$

Use $\phi = c_0 + \sum_{i=1}^k c_i \phi_i$ as a test function for $\lambda_k(f)$. The rest part of the proof is similar to that of the above proposition. \square

Proposition 3.6. *Notations as above. We have*

$$\lambda_k(f) \geq \lambda_{k-1}. \quad (43)$$

Proof. Let $\{\phi_{i,f}\}_{i=1}^k$ be the first k f -eigenfunctions. Then there exist some non-trivial constants $\{c_i\}_{i=1}^k$ such that

$$\sum_{i=1}^k c_i \phi_{i,f} \perp \text{Span}\{1, \{\phi_i\}_{i=1}^{k-2}\}.$$

Taking $\sum_{i=1}^k c_i \phi_{i,f}$ to be a test function of λ_{k-1} , we can obtain the conclusion as in the previous two propositions. \square

Combing Proposition 3.5 and Proposition 3.6, we have proved the following

Theorem 3.7 (Interlace Theorem).

$$0 < \lambda_1(f) \leq \lambda_1 \leq \lambda_2(f) \leq \lambda_2 \leq \cdots \leq \lambda_k(f) \leq \lambda_k \leq \cdots \rightarrow +\infty. \quad (44)$$

Corollary 3.8. *Let $f, h : M^n \rightarrow \mathbb{R}^+$ be positive functions on M^n . Then $\lambda_k(f) \leq \lambda_{k+1}(h)$.*

Below is a sharp estimation of $\lambda_k(f)$.

Proposition 3.9. *Let M be a closed Riemannian manifold and $f : M \rightarrow \mathbb{R}^+$ a positive function. Using the same notations as above, we have*

$$\lambda_k \leq \lambda_k(f) + R(f). \quad (45)$$

Proof. Let $\{\phi_{i,f}\}_{i=1}^k$ be the first k f -eigenfunctions (orthogonal to each other). There exist some non-trivial constants $\{c_i\}_{i=0}^k$ such that

$$c_0 f + \sum_{i=1}^k c_i \phi_{i,f} \perp \text{Span}\{1, \{\phi_i\}_{i=1}^{k-1}\}.$$

As a preparation, by Cauchy-Schwarz inequality we obtain

$$\begin{aligned} (\|c_0 \nabla f\| + \|\sum_{i=1}^k c_i \nabla \phi_{i,f}\|)^2 &= (\|c_0 f\| \cdot \frac{\|\nabla f\|}{\|f\|} + \|\sum_{i=1}^k c_i \phi_{i,f}\| \frac{\|\sum_{i=1}^k c_i \nabla \phi_{i,f}\|}{\|\sum_{i=1}^k c_i \phi_{i,f}\|})^2 \\ &\leq (\|c_0 f\|^2 + \|\sum_{i=1}^k c_i \phi_{i,f}\|^2) \left(\frac{\|\nabla f\|^2}{\|f\|^2} + \frac{\|\sum_{i=1}^k c_i \nabla \phi_{i,f}\|^2}{\|\sum_{i=1}^k c_i \phi_{i,f}\|^2} \right). \end{aligned} \quad (46)$$

So we have

$$\begin{aligned} \lambda_k \leq R(c_0 f + \sum_{i=1}^k c_i \phi_{i,f}) &= \frac{\|c_0 \nabla f + \sum_{i=1}^k c_i \nabla \phi_{i,f}\|^2}{\|c_0 f + \sum_{i=1}^k c_i \phi_{i,f}\|^2} \\ &\leq \frac{(\|c_0 \nabla f\| + \|\sum_{i=1}^k c_i \nabla \phi_{i,f}\|)^2}{\|c_0 f\|^2 + \|\sum_{i=1}^k c_i \phi_{i,f}\|^2} \\ &\leq \frac{\frac{\|\nabla f\|^2}{\|f\|^2} + \frac{\|\sum_{i=1}^k c_i \nabla \phi_{i,f}\|^2}{\|\sum_{i=1}^k c_i \phi_{i,f}\|^2}}{1} \\ &= R(f) + \frac{\sum_{i=1}^k c_i^2 \lambda_i(f) \|\phi_{i,f}\|^2}{\sum_{i=1}^k c_i^2 \|\phi_{i,f}\|^2} \\ &\leq R(f) + \lambda_k(f). \quad \square \end{aligned}$$

4 Counterexamples to the conjecture

Despite our effort, the estimation of $\lambda_1(f)$ we obtained in the previous section is not enough to prove Conjecture 1.2. Naturally one is led to doubt this conjecture and search for counterexamples. Soon after considering this possibility seriously we found such examples.

Before giving the details, let us explain the basic idea at first. Suppose there exists a counterexample to Conjecture 1.2, which is a triple (M, f, ϕ) consisting of a closed Riemannian manifold (M, g) , a positive function $f : M \rightarrow \mathbb{R}^+$, and a smooth function $\phi : M \rightarrow \mathbb{R}$ such that $f \perp \mathcal{U}_1$, $\phi \perp f$ and $\int_M |\nabla \phi|^2 d\mu < \lambda_1 \int_M \phi^2 d\mu$. We would like it to be as simple as possible. So

1. We consider M to be the unit sphere S^n or a flat torus with their standard metrics, because they have many symmetries, and their eigenvalues $\{0 < \lambda_1 < \dots\}$ and eigenfunctions $\{\phi_k\}(\phi_0 \equiv 1)$ are known already. (Here we count the multiplicities of the eigenvalues. This convention is different from the previous section.) Then we can expand any function using the eigenfunctions.

2. We do not fix the positive function f beforehand, partly because its positivity is hard to characterize in terms of its Fourier series. Instead we begin with $\phi = \sum a_k \phi_k$. Because the higher order eigenfunctions ϕ_k have positive contribution to the Rayleigh quotient $R(\phi)$, we would like most of the a_k vanish so that $R(\phi)$ is smaller and the expression $\phi = \sum a_k \phi_k$ is simpler. The best choice is $\phi = \epsilon + \phi_2$ where the constant ϵ will be chosen later.

3. The positive function f is found in the last step. To guarantee $f \perp \phi$, ϕ should change its sign on M , which is also a sufficient condition to guarantee the existence of a positive function f orthogonal to ϕ . So ϵ must be small enough. To have $f \perp \mathcal{U}_1$, it is sufficient for f to be an even function, because on S^n and square torus the first eigenspace is spanned by odd functions. By symmetry, we need only to find a positive f on a fundamental domain U (a quadrant) of M so that $\int_U f \phi d\mu = 0$ and extend it to be an even function on M .

Counterexample 1. Let $S^n = \{(x_1, x_2, \dots, x_{n+1}) \in \mathbb{R}^{n+1} | x_1^2 + x_2^2 + \dots + x_{n+1}^2 = 1\}$ be the sphere with the standard induced metric. It is well-known that $\lambda_1(S^n) = n$, $\lambda_2(S^n) = 2n + 2$, and $\phi_2 = x_1^2 - x_2^2$ is one of the second eigenfunctions which takes value in $[-1, 1]$.

Let $\phi = \epsilon + x_1^2 - x_2^2$ on S^n . There should be $\epsilon \in (-1, 1)$ so that ϕ changes sign. It is easy to see

$$R(\phi) < \lambda_1 \iff \left(\frac{\lambda_2}{\lambda_1} - 1 \right) \frac{\int_{S^n} (\phi_2)^2 d\mu}{\text{Vol}(S^n)} < \epsilon^2. \quad (47)$$

By direct computation,

$$\left(\frac{\lambda_2}{\lambda_1} - 1 \right) \frac{\int_{S^n} (x_1^2 - x_2^2)^2 d\mu}{\text{Vol}(S^n)} = \frac{4(n+2)}{n(n+1)(n+3)}. \quad (48)$$

When $n \geq 2$, $c_n \triangleq \frac{4(n+2)}{n(n+1)(n+3)} < 1$, thus we can take $\epsilon \in (\sqrt{c_n}, 1)$, which guarantees that $R(\phi) < \lambda_1$ and ϕ changes its sign. By the analysis above, this shows that Conjecture 1.2 does not hold for higher dimensional sphere S^n when $n \geq 2$.

Counterexample 2. For the square torus $T^2 = S^1 \times S^1$, we know $\lambda_1(T^2) = 1$, $\lambda_2(T^2) = 2$, and $\phi_2 = \cos x \cos y$ is a second eigenfunction taking value in $[-1, 1]$.

Let $\phi = \epsilon + \cos x \cos y$ on T^2 . By direct computation,

$$\left(\frac{\lambda_2}{\lambda_1} - 1\right) \frac{\int_{T^2} (\phi_2)^2 d\mu}{\text{Vol}(T^2)} = \left(\frac{\lambda_2}{\lambda_1} - 1\right) \frac{\int_0^{2\pi} \int_0^{2\pi} \cos^2 x \cos^2 y dx dy}{4\pi^2} = \frac{1}{4}. \quad (49)$$

Thus we need only to take $\epsilon \in (\frac{1}{2}, 1)$. Then $R(\phi) < \lambda_1$ and ϕ changes its sign. By the same reason as above, we know that Conjecture 1.2 is not true for the square torus.

Remark 4.1. Although our Conjecture 1.2 is generally not true, by Proposition 3.3 one can see that when u is orthogonal to a fixed function f and $\int_M f d\mu \neq 0$, there exists some universal constant C such that the Poincaré type inequality

$$C \int_M u^2 d\mu \leq \int_M |\nabla u|^2 d\mu.$$

still holds.

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